PERGAMON

International Journal of Solids and Structures 39 (2002) 5813–5835

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Separation phenomena at the interface of a finitely deformed composite sphere

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Received 21 July 2001; received in revised form 18 May 2002

Abstract

The problem of the finite deformation of a composite sphere subjected to a spherically symmetric dead load traction is revisited focusing on the formation of a cavity at the interface between a hyperelastic, incompressible matrix shell and a rigid inhomogeneity. Separation phenomena are assumed to be governed by a vanishingly thin interfacial cohesive zone characterized by uniform normal and tangential interface force–separation constitutive relations. Spherically symmetric cavity shapes (spheres) are shown to be solutions of an interfacial integral equation depending on the strain energy density of the matrix, the interface force constitutive relation, the dead loading and the volume concentration of inhomogeneity. Spherically symmetric and non-symmetric bifurcations initiating from spherically symmetric equilibrium states are analyzed within the framework of infinitesimal strain superimposed on a given finite deformation. A simple formula for the dead load required to initiate the non-symmetrical rigid body mode is obtained and a detailed examination of a few special cases is provided. Explicit results are presented for the Mooney–Rivlin strain energy density and for an interface force–separation relation which allows for complete decohesion in normal separation.

Keywords: Non-linear elasticity; Bifurcation problem; Cavity nucleation; Interfacial debonding and decohesion; Inclusion problem

1. Introduction

This paper is concerned with separation phenomena at the interface of a finitely deformed composite sphere. (A recent, related paper (Levy, 2001) has treated the problem of separation of a rigid cylindrical fiber embedded in an unbounded incompressible matrix.) The purpose is to examine the influence of factors such as interface constitutive parameters, volume concentration of inhomogeneity, etc. on the formation of spherically symmetric and non-symmetric cavities by interfacial separation. We adopt the viewpoint, first espoused by Needleman (1987), that atomistically sharp, compliant interfaces can be modeled by uniform cohesive zones of vanishing thickness. Defect dominated response can be incorporated within the cohesive zone framework (Needleman, 1990a,b) although this is not considered here. The first part of the paper

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0020-7683/02/$ - see front matter
PII: S0020-7683(02)00377-3

treats the spherically symmetric problem and essentially extends previous work (Horgan and Pence, 1989a,b) on the finitely deformed, rigid interface composite sphere to configuration dependent (interface) boundary conditions. The focus here however is on interfacial separation phenomena and not on cavitation occurring for example at the center of a uniform elastic inclusion. Problems of this type have received detailed treatment beginning with the work of Ball (1982).

In the first part of the paper, we formulate the equations for spherically symmetric equilibrium although we defer an analysis of the spherically symmetric bifurcation problem to the section that follows where it is treated as a special case. In particular, we derive an integral equation governing the evolution of interface stretch with dead load, which involves the matrix strain energy density, the interface force constitutive law and the volume concentration of inhomogeneity. The second part of the paper examines spherically symmetric and non-symmetric bifurcation from the spherically symmetric equilibrium states considered in the first part. We do this by utilizing the theory of infinitesimal deformation superimposed on a given finite strained state in a manner similar to that of Ogden (1984) in his study of the related problem of bifurcation of the finitely deformed spherical shell subjected to internal pressure. First, a local bifurcation analysis is carried out, assuming superimposed states are spherically symmetric. Global results are presented for the case of the Mooney–Rivlin matrix and an exponential interface force–separation law (Ferrante et al., 1982). Next we focus on the first non-symmetrical mode characterized by the rigid displacement of the inclusion and we produce formulae for the critical load assuming (i) a smooth interface (i.e., only normal tractions are supported) and, (ii) an existing interface constitutive model accounting for interfacial shear-slip. Because the bounded matrix shell, as opposed to unbounded matrix, accounts in some sense for inclusion interaction, results are provided which indicate the influence of volume concentration of inhomogeneity on behavior. We conclude the paper with a consideration of parameter domains that govern the different types of bifurcation.

2. Spherically symmetric equilibrium states

2.1. General formulation

Consider a composite sphere B consisting of an inner spherical domain Ω (the inclusion) and an outer spherical shell B–Ω (the matrix). Assume Ω is bounded by the surface $\partial\Omega^-$ and B–Ω is bounded by outer surface ∂B and inner surface $\partial\Omega^+$, such that in the undeformed state $\partial\Omega^- = \partial\Omega^+$ which we call the interface. Fix a single Cartesian system with origin $\mathbf{o}$ at the inclusion center, material point coordinates (p_1, p_2, p_3), place coordinates (x_1, x_2, x_3) and basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Introduce a spherical coordinate system with coordinates (R, Θ, Φ) and physical basis $(\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi)$ associated with material points, and another spherical coordinate system with coordinates (r, θ, φ) and physical basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ associated with places such that $\mathbf{e}_R(\Theta = 0, \Phi) = \mathbf{e}_r(\theta = 0, \varphi) = \mathbf{e}_3$. The domains Ω and B–Ω are defined by,

$$\begin{aligned} \Omega &= \{(R, \Theta, \Phi) | R \in (0, R_0), \Theta \in (0, \pi), \Phi \in (0, 2\pi)\} \\ B\text{–}\Omega &= \{(R, \Theta, \Phi) | R \in (R_0, R_1), \Theta \in (0, \pi), \Phi \in (0, 2\pi)\}. \end{aligned} \tag{1}$$

In this paper we assume that the inclusion is rigid and that the matrix is isotropic, incompressible and hyperelastic. Furthermore, in this section we seek only spherically symmetric solutions so that we will in effect constrain the inclusion against rigid displacement. This assumption will be relaxed in the section on non-symmetrical solutions that follows.

Spherically symmetric deformations are of the form,

$$r = f(R), \quad \theta = \Theta, \quad \varphi = \Phi, \tag{2}$$

with displacement vector $\mathbf{u}$, deformation gradient tensor $\mathbf{F}$, and left Cauchy–Green strain tensor $\mathbf{B}(=\mathbf{FF}^{\mathbf{T}})$, given by,

$$\begin{aligned}
&\mathbf{u} = (f(R) - R)\mathbf{e}_R, \\
&\mathbf{F} = f'\mathbf{e}_r \otimes \mathbf{e}_R + fR^{-1}\mathbf{e}_\theta \otimes \mathbf{e}_\Theta + fR^{-1}\mathbf{e}_\varphi \otimes \mathbf{e}_\Phi, \\
&\mathbf{B} = (f')^2\mathbf{e}_r \otimes \mathbf{e}_r + (fR^{-1})^2\mathbf{e}_\theta \otimes \mathbf{e}_\theta + (fR^{-1})^2\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi,
\end{aligned} \tag{3}$$

where $[\bullet]'$ indicates differentiation with respect to R. Matrix incompressibility $\det \mathbf{F} = \det \mathbf{B} = 1$ follows from (3) so that,

$$R^{-2}f^2f' = 1. \tag{4}$$

The principal stretches $\lambda_R = |\mathbf{Ue}_R|$, $\lambda_\Theta = |\mathbf{Ue}_\Theta|$, $\lambda_\Phi = |\mathbf{Ue}_\Phi|$ are given by,

$$\lambda_R = \lambda^{-2} = f' = (R/r)^2, \quad \lambda_\Theta = \lambda = R^{-1}f = R^{-1}r, \quad \lambda_\Phi = \lambda = R^{-1}f = R^{-1}r, \tag{5}$$

where $\mathbf{U}(=\sqrt{\mathbf{F}^{\mathbf{T}}\mathbf{F}})$ is the right stretch tensor and use has been made of (4). Integration of the incompressibility constraint (4) yields the deformation and principal stretches,

$$\lambda = (1 + C^2R^{-3})^{1/3}, \quad \lambda \geqslant 1, \tag{6}$$

where the constant in (6) is chosen to be positive so that $r = f(R) > R$ for all $R \in (R_0, R_1)$.

The isotropic, incompressible hyperelastic stress–strain relation for the matrix is, $\mathbf{T} = \mathbf{F}D\hat{\sigma}(\mathbf{F})^{\mathbf{T}} - \hat{\pi}\mathbf{1}$, where $\mathbf{T}$ is the Cauchy stress, $\hat{\sigma}$ is the strain energy density, $\hat{\pi}$ is hydrostatic pressure and $\mathbf{1}$ is the unit tensor. The symbol D indicates differentiation of the function that follows it with respect to its argument. Physical components are given by,

$$T_{rr} = \lambda_R\frac{\partial\hat{\sigma}}{\partial\lambda_R} - \hat{\pi}, \quad T_{\theta\theta} = \lambda_\Theta\frac{\partial\hat{\sigma}}{\partial\lambda_\Theta} - \hat{\pi}, \quad T_{\varphi\varphi} = \lambda_\Phi\frac{\partial\hat{\sigma}}{\partial\lambda_\Phi} - \hat{\pi}. \tag{7}$$

Following Abeyaratne and Horgan (1985) we use (4) to write the single non-trivial equilibrium equation in the form,

$$\frac{\partial T_{rr}}{\partial R} + \frac{2f'}{f}T_{rr} - \frac{f'}{f}(T_{\theta\theta} + T_{\varphi\varphi}) = 0. \tag{8}$$

The boundary condition at the outer surface of the composite sphere consists of a uniform dead load traction applied at $R = R_1$ i.e., $\mathbf{t}(\mathbf{e}_R) = \sigma\mathbf{e}_R$, where $\mathbf{t}(\mathbf{e}_R)$ is the Piola–Kirchoff traction vector (per unit area in the reference state) and σ is assumed to be positive. The Piola–Kirchoff stress ($\mathbf{S}$) is related to the Cauchy stress ($\mathbf{T}$) by, $\mathbf{S} = (\det \mathbf{F})\mathbf{TF}^{-T}$, and the traction $\mathbf{t}(\mathbf{e}_R)$ is related to the stress $\mathbf{S}$ by, $\mathbf{t}(\mathbf{m}) = \mathbf{Sm}$, $\mathbf{m}$ unit normal to undeformed surface. We can therefore write the boundary condition in terms of the radial Cauchy stress component T_{rr}. Because the deformation is spherically symmetric, $\mathbf{e}_r = \mathbf{e}_R$ and it follows that,

$$T_{rr} = \sigma\left(\frac{r_1}{R_1}\right)^{-2} = \sigma\lambda_1^{-2}. \tag{9}$$

The interface boundary condition on the inner surface of the matrix is the traction $\mathbf{s}_\mathrm{I}(\mathbf{n})$ (per unit area in the current configuration), $\mathbf{s}_\mathrm{I}(-\mathbf{e}_r) = -s_r^0\mathbf{e}_r$ on $r_0 = f(R_0)$, which may be expressed as a condition on the radial component of Cauchy stress,

$$T_{rr} = s_r^0, \text{ on } r_0 = f(R_0). \tag{10}$$

The quantity s_r^0 appearing in (10) is the normal interface traction, which defines the constitutive characteristics of the interface. In this paper, we will assume that it is a uniform function of interfacial separation $(r_0 - R_0)$ only, i.e., it is independent of interface coordinate (θ, φ) explicitly.

Following Levy (2001) we employ the equilibrium equation (8) and boundary conditions (9) and (10) to determine the pressure function $\hat{\pi}$ and the interface stretch $\lambda_0(=r_0/R_0)$ that, by (6), fixes the constant C and determines the deformation (2). Equilibrium equation (8) may be written, with the aid of (4) and (7), in a form that may be directly integrated to yield the pressure function,

$$\hat{\pi} = \lambda^{-2}\hat{\sigma}_1(\lambda^{-2},\lambda,\lambda) - \int \frac{D\hat{w}(\lambda)}{2(1-\lambda^3)}\mathrm{d}\lambda + K, \tag{11}$$

where K is constant and where we have employed the standard notation, $\hat{\sigma}_1(\lambda^{-2},\lambda,\lambda) = \partial\hat{\sigma}/\partial\lambda_1$, $\hat{\sigma}_2(\lambda^{-2},\lambda,\lambda) = \hat{\sigma}_3(\lambda^{-2},\lambda,\lambda) = \partial\hat{\sigma}/\partial\lambda_2$. For compactness we have introduced the strain energy function defined by $\hat{w}(\lambda) = \hat{\sigma}(\lambda^{-2},\lambda,\lambda)$ so that,

$$\lambda D\hat{w}(\lambda) = 2[\lambda\hat{\sigma}_2(\lambda^{-2},\lambda,\lambda) - \lambda^{-2}\hat{\sigma}_1(\lambda^{-2},\lambda,\lambda)], \tag{12}$$

which has been used in (11).

The equations governing the interface stretch λ_0 and the constant K are determined from boundary conditions (9) and (10) and pressure (11) and are given by,

$$0 = F(\lambda_0,\sigma) = -\sigma + [1+c(\lambda_0^3-1)]^{2/3}\left\{s_r^0(\lambda_0) - \int_{[1+c(\lambda_0^3-1)]^{1/3}}^{\lambda_0} \frac{D\hat{w}(\lambda)}{(1-\lambda^3)}\mathrm{d}\lambda\right\}, \quad \lambda_0 \geqslant 1, \tag{13a}$$

$$K = -\lambda_1^{-2}\sigma + \int^{\lambda_1} \frac{D\hat{w}(\lambda)}{(1-\lambda^3)}\mathrm{d}\lambda = -s_r^0(\lambda_0) + \int^{\lambda_0} \frac{D\hat{w}(\lambda)}{(1-\lambda^3)}\mathrm{d}\lambda, \tag{13b}$$

where c is the volume concentration of inhomogeneity defined to be,

$$c = \left(\frac{R_0}{R_1}\right)^3. \tag{14}$$

Note that in (13a) and (13b) we have taken the interface stretch $\lambda_0(=r_0/R_0)$ as a measure of the interface separation $(r_0 - R_0)$ and we have expressed λ_1 in terms of λ_0 by virtue of (6), i.e.,

$$\lambda_1 = [1+c(\lambda_0^3-1)]^{1/3}, \quad \lambda_1 \in [1,\lambda_0). \tag{15}$$

Eq. (13a) determines the evolution of normalized interface separation or displacement jump $\lambda_0 - 1(=u_r(R_0)/R_0)$ with load σ and depends on the matrix strain energy density $\hat{\sigma}$ (or $\hat{w}$) the interface force law s_r^0, and the volume concentration c. The stretch then follows from (6),

$$\lambda = \left[1 + \left(\frac{R_0}{R}\right)^3(\lambda_0^3-1)\right]^{1/3}, \tag{16}$$

The solution of (13a) determines the constant K by (13b) which determines the pressure $\hat{\pi}$ from (11),

$$\begin{aligned}\hat{\pi} &= \lambda^{-2}\hat{\sigma}_1(\lambda^{-2},\lambda,\lambda) - \int_{\lambda_1}^{\lambda} \frac{D\hat{w}(t)}{(1-t^3)}\mathrm{d}t - \lambda_1^{-2}\sigma \\ &= \lambda^{-2}\hat{\sigma}_1(\lambda^{-2},\lambda,\lambda) + \int_{\lambda}^{\lambda_0} \frac{D\hat{w}(t)}{(1-t^3)}\mathrm{d}t - s_r^0(\lambda_0),\end{aligned} \tag{17}$$

and the stress components T_{rr}, $T_{\theta\theta}$, $T_{\varphi\varphi}$ from (7),

$$
\begin{aligned}
T_{rr} &= \int_{\lambda_1}^{\lambda} \frac{D\hat{w}(t)}{(1-t^3)}\,\mathrm{d}t + \lambda_1^{-2}\sigma = -\int_{\lambda}^{\lambda_0} \frac{D\hat{w}(t)}{(1-t^3)}\,\mathrm{d}t + s_r^0(\lambda_0), \\
T_{\theta\theta} &= T_{\varphi\varphi} = \frac{\lambda}{2} D\hat{w}(\lambda) + T_{rr}.
\end{aligned} \tag{18}
$$

A discussion of the behavior predicted by (13a) and (18), for a specific strain energy density and interface force law, is deferred to the next section after a treatment of the spherically symmetric bifurcation problem.

3. Spherically symmetric initially strained equilibrium states

3.1. Governing equations

In what follows, we consider three distinct states, the undeformed initial state, the spherically symmetric finitely deformed reference state, and the non-symmetrical current state (obtained from the reference state by an infinitesimal deformation). Because we have assumed that there is no directional bias in the geometry or loading we can, without loss of generality, assume that non-symmetrical configurations are such that all fields will be independent of longitudinal angle φ. This assumption is consistent with a rigid inclusion displacement in the $\mathbf{e}_3$ direction. The spherical components of the infinitesimal displacement gradient tensor $\mathbf{H}(=\nabla\mathbf{u})$ consistent with this deformation are given by,

$$
[\mathbf{H}] = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & r^{-1}\left(\dfrac{\partial u_r}{\partial \theta} - u_\theta\right) & 0 \\ \dfrac{\partial u_\theta}{\partial r} & r^{-1}\left(\dfrac{\partial u_\theta}{\partial \theta} + u_r\right) & 0 \\ 0 & 0 & r^{-1}(u_r + u_\theta\cot\theta) \end{bmatrix}, \tag{19}
$$

where $\mathbf{u}$ is the infinitesimal displacement field dependent on spherical coordinates (r, θ, φ) which identify places in the reference configuration. The tensor $\mathbf{H}$ is subject to the incompressibility constraint $\mathrm{tr}\,\mathbf{H} = 0$. Let $\Delta\mathbf{S}$ be the incremental Piola–Kirchoff stress (force per unit area in the reference state) supporting the infinitesimal displacement field $\mathbf{u}$. The incremental stress–displacement relations follow from the general theory of infinitesimal strain superimposed on a given finite strain and the non-zero spherical components may be written in the form,

$$
\begin{aligned}
\Delta S_{rr} &= -\Delta\pi + \beta_1 \frac{\partial u_r}{\partial r}, \\
\Delta S_{\theta\theta} &= -\Delta\pi + \beta_2 \frac{\partial u_r}{\partial r} + \beta_3 r^{-1}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \\
\Delta S_{\varphi\varphi} &= -\Delta\pi + (\beta_2 - \beta_3)\frac{\partial u_r}{\partial r} - \beta_3 r^{-1}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \\
\Delta S_{r\theta} &= \beta_5 r^{-1}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \beta_4 \frac{\partial u_\theta}{\partial r}, \\
\Delta S_{\theta r} &= \beta_4 r^{-1}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \beta_6 \frac{\partial u_\theta}{\partial r},
\end{aligned} \tag{20}
$$

where $\Delta\pi$ is the incremental hydrostatic pressure and the β_i coefficients depend on the finite deformation through λ. Note that from (5) and (16) λ may be written as a function of coordinate r,

$$
\lambda = \left[1 - \left(\frac{r_0}{r}\right)^3\left(1 - \frac{1}{\lambda_0^3}\right)\right]^{-1/3}. \tag{21}
$$

Displacement equations of equilibrium are obtained when (20) is substituted into the spherical components of the incremental equilibrium equation $\mathrm{div}(\Delta\mathbf{S}) = \mathbf{0}$. The two non-trivial equilibrium equations and the incompressibility equation are given by,

$$\begin{gathered}\frac{\partial\Delta\pi}{\partial r} - (\beta_1 - \beta_4)\frac{\partial^2 u_r}{\partial r^2} - \beta_5 r^{-2}\left(\frac{\partial^2 u_r}{\partial\theta^2} + \cot\theta\frac{\partial u_r}{\partial\theta} + 2u_r\right) - (r\beta_1' + 2\beta_1 - 2\beta_2 + \beta_3 - 3\beta_4 + \beta_5)r^{-1}\frac{\partial u_r}{\partial r} = 0,\\ r^{-1}\frac{\partial\Delta\pi}{\partial\theta} - \beta_6\frac{\partial^2 u_\theta}{\partial r^2} - (\beta_2 + \beta_4 - \beta_3)r^{-1}\frac{\partial^2 u_r}{\partial r\partial\theta} - (r\beta_4' - \beta_3 + \beta_4 + \beta_5)r^{-2}\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right) - (r\beta_6' + 2\beta_6)r^{-1}\frac{\partial u_\theta}{\partial r} = 0,\\ \frac{\partial u_r}{\partial r} + r^{-1}\left(\frac{\partial u_\theta}{\partial\theta} + 2u_r + \cot\theta u_\theta\right) = 0, \end{gathered} \tag{22}$$

Note that we now employ the notation $[\bullet]'$ to indicate the derivative with respect to r. Ogden (1984) has obtained these equations in his analysis of the bifurcation of a pressurized spherical shell. The linearity of Eq. (22) suggests that a solution be sought in the form of an eigenfunction expansion. Because we have assumed that the superimposed non-spherically symmetric equilibrium state is symmetric with respect to any longitudinal plane the radial displacement u_r and incremental pressure $\Delta\pi$ are chosen to be even functions of θ while angular displacement u_θ is chosen to be an odd function of θ. Thus, we represent the solution in the form of an expansion of Legendre polynomials $P_n(\cos\theta)$,

$$\begin{aligned} u_r &= U_0(r) + \sum_{n=1}^{N} U_n(r)P_n(\cos\theta),\\ u_\theta &= \sum_{n=1}^{N} V_n(r)\dot{P}_n(\cos\theta),\\ \Delta\pi &= \Pi_0(r) + \sum_{n=1}^{N} \Pi_n(r)P_n(\cos\theta), \end{aligned} \tag{23}$$

where $\dot{P}_n(\cos\theta) = \mathrm{d}P_n(\cos\theta)/\mathrm{d}\theta$ and $P_1(\cos\theta) = \cos\theta$. Ogden (1984) has shown that the substitution of (23) into (22) leads to a single ordinary differential equation of fourth order governing mode multipliers $U_n(r)$, $n = 1, 2, \ldots, N$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r}\{\beta_6 r^4 U_n''' + [r\beta_6' + 4\beta_6]r^3 U_n'' + [n(n+1)(\beta_2 + 2\beta_4 - \beta_3 - \beta_1) - r\beta_4' + 3r\beta_6' + \beta_3 - \beta_4 - \beta_5]r^2 U_n'\}\\ &\quad - (n^2 + n - 2)\left\{r\beta_4' - \beta_3 + \beta_4 + \beta_5 - n(n+1)\beta_5 - r\frac{\mathrm{d}}{\mathrm{d}r}(r\beta_4' - \beta_3 + \beta_4 + \beta_5)\right\}U_n = 0, \quad n = 1, 2, \ldots, N. \end{aligned} \tag{24}$$

The mode multipliers $\Pi_n(r)$, $V_n(r)$ in turn are obtained from,

$$\Pi_n' = (\beta_1 - \beta_4)U_n'' + (r\beta_1' + 2\beta_1 - 2\beta_2 + \beta_3 - 3\beta_4 + \beta_5)r^{-1}U_n' + \beta_5 r^{-2}[2 - n(n+1)]U_n, \tag{25a}$$

$$V_n = \frac{1}{n(n+1)}(rU_n' + 2U_n), \quad n = 1, 2, \ldots, N. \tag{25b}$$

The spherically symmetric ($n = 0$) mode multipliers $U_0(r)$, $\Pi_0(r)$ follow directly from (22),

$$\begin{aligned} \Pi_0 &= K_2 - 2K_1\int(r\beta_1' - \beta_1 - 2\beta_2 + \beta_3)r^{-4}\mathrm{d}r,\\ U_0 &= K_1 r^{-2}, \end{aligned} \tag{26}$$

where K_1, K_2 are constants to be determined from external and interface boundary conditions. (Note that $V_0 = 0$ since u_θ is an odd function of θ.) Eq. (24) can be integrated exactly for the $n = 1$ mode (Ogden, 1984). In our notation the result is,

$$rU_1'' + (\lambda^3 + 3)U_1' = \frac{C_1 r}{\lambda\beta_6}, \quad r^4 U_1' = 2C_1\lambda \int^r t^4 \frac{\lambda^3 - \lambda^{-3}}{D\hat{w}(\lambda)} \mathrm{d}t + \lambda C_2, \tag{27}$$

where we have made use of the relation,

$$\beta_6 = \frac{\lambda}{2(\lambda^6 - 1)} D\hat{w}(\lambda). \tag{28}$$

β_6 we take to be positive by virtue of the Baker–Ericksen inequality, i.e.,

$$\frac{D\hat{w}(\lambda)}{\lambda^3 - 1} = \frac{2\lambda}{(\lambda^3 - 1)^2}[\lambda_\Theta - \lambda_R][T_{\theta\theta} - T_{rr}] > 0, \quad \lambda \geqslant 1, \tag{29}$$

where use has been made of (12) and (18).

3.2. Boundary conditions

The boundary conditions consist of a dead load traction on the outer surface of the sphere and a configuration dependent interface condition at the inner boundary of the matrix shell. The dead loading at $r = r_1$ is an incremental, equibiaxial traction, $\Delta\mathbf{t}(\mathbf{e}_r) = \Delta\mathbf{S}\mathbf{e}_r = \Delta\sigma\mathbf{e}_r$, where $\Delta\sigma$ is assumed positive. This condition can be stated in terms of the displacements and their derivatives,

$$-\Delta\pi + \beta_1 \frac{\partial u_r}{\partial r} = \Delta\sigma, \quad r = r_1, \tag{30a}$$

$$(\beta_6 - \lambda_1^{-2}\sigma) r^{-1}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \beta_6 \frac{\partial u_\theta}{\partial r} = 0, \quad r = r_1, \tag{30b}$$

where use has been made of (20) and it is understood that β_i are evaluated at $\lambda = \lambda_1$. Note that in obtaining (30b) we have made use of the relation, $\beta_6(\lambda_1) = \beta_4(\lambda_1) + \lambda_1^{-2}\sigma$, where σ is the applied equibiaxial dead load supporting the finitely deformed spherically symmetric reference state.

Appropriate measures of the displacement jump at the interface have been determined in Levy (2001), which enables the formulation of the interface boundary condition. (Recall that the inclusion may now displace rigidly in the superimposed infinitesimal deformation and, because of this, the interface force no longer depends on the inner boundary displacement of the matrix but on the relative displacement of the matrix boundary and inclusion.) Here we reiterate some of these results. The infinitesimal rigid body translation of the inclusion in the $\mathbf{e}_3$ coordinate direction is, $\mathbf{u}_0 = w_0\mathbf{e}_3 = w_0(P_1(\cos\theta)\mathbf{e}_R + \dot{P}_1(\cos\theta)\mathbf{e}_\Theta)$, where w_0 is constant. Note that we are considering rigid displacements that leave the inclusion non-rotated. Write $u_R\mathbf{e}_R = (r_0 - R_0)\mathbf{e}_R$, for the initial spherically symmetric finite displacement of points $\mathbf{p_0}$ on the inner boundary of the matrix to points $\mathbf{x_0}$ in the reference state. The quantity $r_0(= f(R_0))$ locates points $\mathbf{x_0}$, which prior to deformation, were at radius R_0. The displacement of points $\mathbf{p_0}$ to points $\tilde{\mathbf{x}}_0(= \mathbf{o} + \tilde{r}_0\mathbf{e}_{\tilde{r}})$ on the current, non-symmetric inner matrix boundary is, $\tilde{\mathbf{u}} = \tilde{u}_r\mathbf{e}_R + \tilde{u}_\theta\mathbf{e}_\Theta = \tilde{r}_0\mathbf{e}_{\tilde{r}} - R_0\mathbf{e}_R$, where $\tilde{r}_0$ locates points which, prior to deformation were at radius R_0. The infinitesimal displacement $\mathbf{u}$ of points $\mathbf{x_0}$ on the inner matrix boundary in the reference state to points $\tilde{\mathbf{x}}_\mathbf{0}$ in the current configuration is given by, $\mathbf{u}(\mathbf{x_0}) = \tilde{\mathbf{x}}_\mathbf{0} - \mathbf{x_0} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta = \tilde{r}_0\mathbf{e}_{\tilde{r}} - r_0\mathbf{e}_r = \tilde{u}_r\mathbf{e}_R + \tilde{u}_\theta\mathbf{e}_\Theta - u_R\mathbf{e}_R$, with r_0 taken as a function of R_0, and $\tilde{r}_0$ is a function of r_0 and θ. Furthermore, $\mathbf{e}_r = \mathbf{e}_R$, $\mathbf{e}_\theta = \mathbf{e}_\Theta$ exactly, because the initial finite deformation is spherically symmetric, but generally, $\mathbf{e}_{\tilde{r}} \neq \mathbf{e}_r$, $\mathbf{e}_{\tilde{\theta}} \neq \mathbf{e}_\theta$. The displacement jump at the inclusion-matrix interface can be written as the difference between the displacement of matrix boundary points in the current configuration and the rigid displacement of the inclusion, $[\tilde{\mathbf{u}}] = \tilde{\mathbf{u}} - \mathbf{u}_0$. For points on the interface the reference stretch and the current stretch are,

$$\lambda_0 = \frac{f(R_0)}{R_0} = \frac{r_0}{f^{-1}(r_0)} = \frac{u_R(R_0)}{R_0} + 1, \tag{31a}$$

$$\tilde{\lambda}_0 = \frac{\tilde{r}_0}{f^{-1}(r_0)} = \frac{\tilde{u}_r(r_0,\theta)}{R_0} + 1, \tag{31b}$$

where we note that the second equality in (31b) holds provided we neglect terms of order $\mathrm{O}(|\mathbf{u}|^2)$, i.e., $\mathbf{u}$ is infinitesimal. Now let u, v, $\tilde{u}$, $\tilde{v}$ represent the normalized variables,

$$\begin{aligned} u &= \frac{u_r - w_0 P_1(\cos\theta)}{r_0}, \quad v = \frac{u_\theta - w_0 \dot{P}_1(\cos\theta)}{r_0}, \\ \tilde{u} &= \frac{\tilde{u}_r - w_0 P_1(\cos\theta)}{R_0}, \quad \tilde{v} = \frac{\tilde{u}_\theta - w_0 \dot{P}_1(\cos\theta)}{R_0}, \end{aligned} \tag{32}$$

where u, v represent incremental interface displacement jump components arising from the superimposed infinitesimal deformation and $\tilde{u}$, $\tilde{v}$ represent the normalized components of displacement jump from the undeformed configuration to the current configuration. Then,

$$\tilde{u}(r_0,\theta) = \lambda(r_0)[1 + u(r_0,\theta)] - 1, \quad \tilde{v}(r_0,\theta) = \lambda(r_0)v(r_0,\theta), \tag{33}$$

where $\mathbf{e}_{\tilde{r}} \cdot \mathbf{e}_r = 1 + \mathrm{O}(|\mathbf{u}|^2)$, $\mathbf{e}_{\tilde{\theta}} \cdot \mathbf{e}_\theta = 1 + \mathrm{O}(|\mathbf{u}|^2)$ and additionally, because the initial finite deformation is spherically symmetric, $u_\theta = \tilde{u}_\theta$.

The interface boundary condition is taken to be a configuration dependent traction $\mathbf{s}_\mathrm{I}(\mathbf{n})$ (force per unit area in the current configuration) given by,

$$\mathbf{s}_\mathrm{I}(-\mathbf{e}_{\tilde{r}}) = -s_r\mathbf{e}_{\tilde{r}} - s_\theta\mathbf{e}_{\tilde{\theta}} \text{ on } \tilde{r}_0 = \tilde{f}(r_0,\theta), \tag{34}$$

where $\mathbf{e}_{\tilde{r}}$, $\mathbf{e}_{\tilde{\theta}}$ are respectively unit vectors normal and tangent to the interface in the current configuration (an additional component s_φ we take to be identically zero owing to the independence the displacement field on longitudinal angle). The normal and shear interface traction components s_r, s_θ, represent the constitutive characteristics of the interface and are functions of the normalized interface displacement jump components $\tilde{u}$, $\tilde{v}$ (note that from (10), $s_r(\lambda_0 - 1, 0) = s_r^0(\lambda_0)$). Additional restrictions placed on the constitutive functions s_r, s_θ are,

$$\begin{aligned} &s_\theta(\tilde{u},0) = 0, \\ &D_{\tilde{v}}s_r(\tilde{u},0) = 0, \\ &D_{\tilde{u}}s_\theta(\tilde{u},0) = 0, \\ &s_r(\tilde{u},\tilde{v}) = s_r(\tilde{u},-\tilde{v}), \\ &s_\theta(\tilde{u},\tilde{v}) = -s_\theta(\tilde{u},-\tilde{v}), \end{aligned} \tag{35}$$

where we note that (35d) implies (35b) and (35e) implies (35a) which implies (35c).

Relations (35) ensure that (i) no interfacial shear force develops in response to spherically symmetric separation, (ii) normal and shear interface force response maintains the physically appropriate sign under the transformation $\tilde{v} \to -\tilde{v}$, (iii) no linear coupling of normal and tangential displacement jump modes. By (33), the displacement jump components can be expressed as functions of u, v that are normalized with respect to r_0 in the reference state. The interface traction (34) is related to the incremental Piola–Kirchoff traction in the reference configuration as follows,

$$\Delta\mathbf{t}(-\mathbf{e}_r) = -\Delta\mathbf{S}\mathbf{e}_r = -\mathbf{s}_\mathrm{I}(\mathbf{e}_r) + \mathbf{s}_\mathrm{I}^\mathbf{0}(\mathbf{e}_r) + \mathbf{s}_\mathrm{I}^\mathbf{0}(\mathbf{H}^\mathrm{T}\mathbf{e}_r), \tag{36}$$

where use has been made of the relation (Truesdell and Noll, 1965), $\mathbf{T} = \mathbf{T}_0(\mathbf{1} + \mathbf{H}^{\mathbf{T}}) + \Delta\mathbf{S} = \mathbf{S} + \mathbf{T}_0\mathbf{H}^{\mathbf{T}}$ and we have written, $\mathbf{s}_{\mathrm{I}}^{\mathbf{0}}(\mathbf{e}_r) = \mathbf{T}_0\mathbf{e}_r$, as the interface traction supporting the reference finite deformation. Implicit in boundary condition (36) is the fact that we have neglected terms of order $\mathrm{O}(|\mathbf{u}|^2)$.

Now the vector $\mathbf{s}_{\mathrm{I}}^{\mathbf{0}}(\mathbf{H}^{\mathbf{T}}\mathbf{e}_r) = s_r^0\mathbf{H}^{\mathbf{T}}\mathbf{e}_r$ may be expanded by utilizing (19) so that the final form of the interface boundary condition may be obtained by combining that result with (34), (36) and (20),

$$\begin{aligned} &[\beta_1(\lambda_0) + s_r^0]\frac{\partial u_r}{\partial r}\bigg|_{r_0,\theta} - [\Delta\pi + (s_r - s_r^0)] = 0, \\ &[\beta_4(\lambda_0) + s_r^0]r_0^{-1}\left[\frac{\partial u_r}{\partial \theta}\bigg|_{r_0,\theta} - u_\theta(r_0,\theta)\right] + \beta_6(\lambda_0)\frac{\partial u_\theta}{\partial r}\bigg|_{r_0,\theta} - s_\theta = 0, \end{aligned} \tag{37}$$

where s_r^0 is the magnitude of $\mathbf{s}_{\mathrm{I}}^0(\mathbf{e}_r)$.

Rigid body inclusion equilibrium must be satisfied as well and these constraints can be written as integrals over the inclusion surface in the undeformed configuration. If we neglect terms of order $\mathrm{O}(|\nabla\mathbf{u}|)$ and $\mathrm{O}(|\mathbf{u}|)$, and recall that s_r, s_θ have been assumed to be independent of longitudinal angle φ, there is only one non-trivial equation,

$$\int_0^\pi [s_r P_1(\cos\theta) + s_\theta \dot{P}_1(\cos\theta)]\sin\theta\,\mathrm{d}\theta = 0. \tag{38}$$

The boundary conditions governing the modes $U_n(r)$, $V_n(r)$, $\Pi_n(r)$ in the expansions (23) follow from (30a), (30b) and (37) and the orthogonality relations for Legendre polynomials and their derivatives,

$$\begin{aligned} &\int_0^\pi P_m(\cos\theta)P_n(\cos\theta)\sin\theta\,\mathrm{d}\theta = \frac{2}{2n+1}\delta_{mn}, \\ &\int_0^\pi \dot{P}_m(\cos\theta)\dot{P}_n(\cos\theta)\sin\theta\,\mathrm{d}\theta = \frac{2n(n+1)}{2n+1}\delta_{mn}. \end{aligned} \tag{39}$$

On the outer surface on the composite sphere they are given by,

$$-\Pi_0(r_1) + \beta_1(\lambda_1)U_0'(r_1) = \Delta\sigma, \tag{40a}$$

$$-\Pi_n(r_1) + \beta_1(\lambda_1)U_n'(r_1) = 0, \quad n = 1,2,\ldots, \tag{40b}$$

$$r_1^{-1}[\beta_6(\lambda_1) - \lambda_1^{-2}\sigma][U_n(r_1) - V_n(r_1)] + \beta_6(\lambda_1)V_n'(r_1) = 0, \quad n = 1,2,\ldots, \tag{40c}$$

while the interface boundary conditions are,

$$-\Pi_0(r_0) + [\beta_1(\lambda_0) + s_r^0]U_0'(r_0) = \frac{1}{2}\int_0^\pi (s_r - s_r^0)\sin\theta\,\mathrm{d}\theta, \tag{41a}$$

$$-\Pi_n(r_0) + [\beta_1(\lambda_0) + s_r^0]U_n'(r_0) = \frac{2n+1}{2}\int_0^\pi (s_r - s_r^0)P_n(\cos\theta)\sin\theta\,\mathrm{d}\theta, \quad n = 1,2,\ldots, \tag{41b}$$

$$r_0^{-1}[\beta_4(\lambda_0) + s_r^0][U_n(r_0) - V_n(r_0)] + \beta_6(\lambda_0)V_n'(r_0) = \frac{2n+1}{2n(n+1)}\int_0^\pi s_\theta \dot{P}_n(\cos\theta)\sin\theta\,\mathrm{d}\theta, \quad n = 1,2,\ldots \tag{41c}$$

Note that it follows from (25a) and (25b) that for $n = 1$ the boundary conditions (40b) and (40c) are identical. This is also true (for $n = 1$) for (41b) and (41c). These can be written in the final form,

$$\beta_6(\lambda_1)r_1 U_1''(r_1) + [2\beta_6(\lambda_1) + \lambda_1^{-2}\sigma]U_1'(r_1) = 0, \tag{42a}$$

$$\beta_6(\lambda_0)r_0U_1''(r_0)+2\beta_6(\lambda_0)U_1'(r_0)=\frac{3}{2}\int_0^\pi s_rP_1(\cos\theta)\sin\theta\,\mathrm{d}\theta. \tag{42b}$$

Before applying the boundary conditions to the first two modes we note that the theory of infinitesimal strain superimposed on a finite deformation enables us to retain only terms linear in u, v (defined in (32)) in the Taylor expansion of $s_r(\tilde{u},\tilde{v})$, $s_\theta(\tilde{u},\tilde{v})$,

$$\begin{aligned} s_r(\tilde{u},\tilde{v}) &= s_r(\lambda_0-1,0)+D_{\tilde{u}}s_r(\lambda_0-1,0)\lambda_0u+\mathrm{O}(u^2),\\ s_\theta(\tilde{u},\tilde{v}) &= D_{\tilde{v}}s_\theta(\lambda_0-1,0)\lambda_0v+\mathrm{O}(v^2). \end{aligned} \tag{43}$$

Neglecting terms of second order, these relations will be used in boundary conditions (42a) and (42b) and the rigid body equilibrium constraint (38) for the remainder of the paper.

3.3. The spherically symmetric bifurcation mode

Application of boundary conditions (40a) and (41a) yields the constants K_1, K_2 which completely determines the spherically symmetric mode $u_r=U_0(r)$, $\Delta\pi=\Pi_0(r)$, $u_\theta=0$ given by (26). (It satisfies identically the rigid body equilibrium equation (38).) By making use of (23), (32) and (39) we obtain,

$$\left[\frac{-\lambda_0^2\lambda_1^{-2}}{(\lambda_0^3-1)}D\hat{w}(\lambda_1)+\frac{1}{(\lambda_0^3-1)}D\hat{w}(\lambda_0)+2c\lambda_0^2\lambda_1^{-5}\sigma+D_{\tilde{u}}s_r(\lambda_0-1,0)\right]\frac{U_0(r_0)}{r_0}=\lambda_0^{-1}\lambda_1^{-2}\Delta\sigma, \tag{44}$$

which governs spherically symmetric bifurcation and continuing spherically symmetric deformation under the incremental load $\Delta\sigma$. Spherically symmetric eigenmodes are obtained from the homogeneous incremental problem formed by setting $\Delta\sigma$ to zero in (44). Thus, non-trivial solutions U_0/r_0 exist provided,

$$\frac{2c\lambda_0^2}{1+c(\lambda_0^3-1)}\sigma+\hat{\lambda}_1(\lambda_0)^2D_{\tilde{u}}s_r(\lambda_0-1,0)-\frac{\lambda_0^2}{(\lambda_0^3-1)}\left[D\hat{w}(\hat{\lambda}_1(\lambda_0))-\left(\frac{\hat{\lambda}_1(\lambda_0)}{\lambda_0}\right)^2D\hat{w}(\lambda_0)\right]=0, \tag{45}$$

where $\hat{\lambda}_1$ is the function defined by (15). Eq. (45) indicates that the slope of the interface force law $D_{\tilde{u}}s_r(\lambda_0-1,0)$ can be positive or negative at bifurcation. Note that (45) is equivalent to $\Delta\lambda_0\,F(\lambda_0,\sigma)=0$ where $F(\lambda_0,\sigma)$ is given by (13a).

In order to explore the local behavior of solutions near the bifurcation points we will assume the Baker–Ericksen inequalities are satisfied so that by (29) $D\hat{w}(\lambda_0)$ is positive. For a given $\hat{w}$ and s_r let solutions to the set (13a) and (45), when they exist, be labeled $(\bar{\lambda},\bar{\sigma})$ where it is understood that there may be more then one pair $(\bar{\lambda},\bar{\sigma})$ which satisfies the equations.

To examine local behavior about a specific bifurcation point expand (13a) in a power series about (bifurcation point) $(\bar{\lambda},\bar{\sigma})$ yielding,

$$F(\lambda_0,\sigma)=-(\sigma-\bar{\sigma})+F(\bar{\lambda},\bar{\sigma})+D_\lambda F(\bar{\lambda},\bar{\sigma})(\lambda_0-\bar{\lambda})+\tfrac{1}{2}D_\lambda^2F(\bar{\lambda},\bar{\sigma})(\lambda_0-\bar{\lambda})^2+\mathrm{O}(3), \tag{46}$$

where O(3) indicates a term of order $(\lambda_0-\bar{\lambda})^3$. The second term on the right-hand side of (46) vanishes by (13a) because $(\bar{\lambda},\bar{\sigma})$ is an equilibrium point. The third term vanishes by (45) because $(\bar{\lambda},\bar{\sigma})$ is a bifurcation point. Local behavior near the bifurcation point is therefore governed by the approximation,

$$\chi-x_\lambda^2=0, \tag{47a}$$

$$\chi=\frac{2(\sigma-\bar{\sigma})}{D_\lambda^2F(\bar{\lambda},\bar{\sigma})},\quad x_\lambda=\lambda_0-\bar{\lambda}, \tag{47b}$$

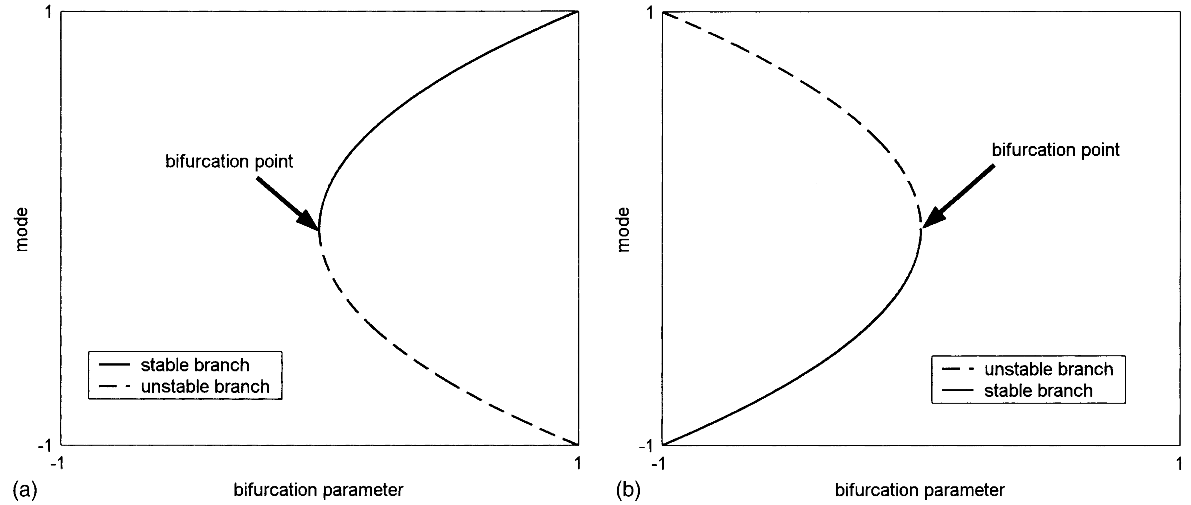

Fig. 1. (a) The saddlenode bifurcation: $D^2F(\bar{\lambda},\bar{\sigma}) > 0$; (b) The saddlenode bifurcation: $D^2F(\bar{\lambda},\bar{\sigma}) < 0$.

where we have neglected the term O(3). If $D^2F(\bar{\lambda},\bar{\sigma}) \neq 0$ then (47a) is the generic form of the saddle node bifurcation from which it follows that local behavior is governed by the sign of the second derivative $D^2F(\bar{\lambda},\bar{\sigma})$. In particular, if $D^2F(\bar{\lambda},\bar{\sigma}) > 0$ then $\sigma - \bar{\sigma} > 0$ and the bifurcation diagram has the form given in Fig. 1(a). If $D^2F(\bar{\lambda},\bar{\sigma}) < 0$ then $\sigma - \bar{\sigma} < 0$ and the bifurcation diagram has the form shown in Fig. 1(b). Note that the broken lines in Fig. 1(a), (b) represent unstable behavior as defined by the energy criterion with energy U defined through its gradient, i.e., $DU = F$. The second derivative $D^2_{\lambda}F(\bar{\lambda},\bar{\sigma})$ may be determined from (13a) but will not be presented since it yields no qualitative information about the behavior. In the following section we will examine the response for a particular matrix strain energy $\hat{w}$ and a specific interface force law s_r. We close this section by considering a number of special cases. Their governing equations are obtained by suitable restriction of (13a) and (45). In particular, we consider the case of an unbounded matrix, the case of a thin matrix shell, and a non-linear infinitesimal strain theory for the composite sphere.

3.3.1. The unbounded matrix

Let R_1 become infinite which, by (14), requires that c vanish. The interface equation (13a) becomes,

$$0 = F(\lambda_0,\sigma) = -\sigma + s_r^0(\lambda_0) + \int_1^{\lambda_0} \frac{D\hat{w}(\lambda)}{(\lambda^3 - 1)} \mathrm{d}\lambda, \quad \lambda_0 \geqslant 1, \tag{48}$$

while the bifurcation condition (45) assumes the form,

$$D_{\lambda_0}s_r^0(\lambda_0) = -\frac{D\hat{w}(\lambda_0)}{(\lambda_0^3 - 1)} = -\frac{2\lambda_0}{(\lambda_0^3 - 1)^2}[\lambda_\Theta - \lambda_R][T_{\theta\theta} - T_{rr}], \tag{49}$$

where use has been made of (29) and we have set $D\hat{w}(1) = 0$ implying that there is no residual stress. Note that by the Baker–Ericksen inequalities we have the result that for the unbounded matrix bifurcation always occurs when $D_{\lambda_0}s_r^0(\lambda_0) < 0$, i.e., the interface force is on the descending branch of the force–separation curve. Eqs. (48) and (49) directly parallel equations obtained in the planar analysis of Levy (2001).

The thin matrix shell. Let the parameter ε be defined by $3(R_1 - R_0)/R_0$ so that the volume concentration $c = 1 - \varepsilon + O(\varepsilon^2)$. The interface equation for this case is obtained by taking a Taylor series of (13a) about $\varepsilon = 0$ and retaining terms linear in ε. The result is,

$$0 = F(\lambda_0, \sigma) = \sigma - \tfrac{1}{3}D\hat{w}(\lambda_0)\varepsilon - \lambda_0^2 s_r^0(\lambda_0). \tag{50}$$

The bifurcation condition follows similarly from (45) or, directly from (50),

$$D_{\lambda_0}s_r^0(\lambda_0) = -2\lambda_0^{-1}s_r^0(\lambda_0) - \tfrac{1}{3}\varepsilon\lambda_0^{-2}D^2\hat{w}(\lambda_0) \tag{51a}$$
$$= -2\lambda_0^{-3}\sigma + \tfrac{2}{3}\varepsilon\lambda_0^{-3}D\hat{w}(\lambda_0) - \tfrac{1}{3}\varepsilon\lambda_0^{-2}D^2\hat{w}(\lambda_0). \tag{51b}$$

It is apparent from (51a) that $D_{\lambda_0}s_r^0(\lambda_0) < 0$ since s_r^0 is always positive for $\lambda_0 > 0$ and the second term on the right-hand side of (51a) is a term of order ε. Thus, is this case bifurcation occurs on the descending branch of the interface force–separation curve as well.

3.3.2. Infinitesimal strain theory

Here we obtain infinitesimal strain forms of interface equation (13a), and bifurcation condition (45). First, assume that the interface force $s_r^0(\lambda_0)$ generally depends on an additional (dimensionless, positive) length parameter ρ, which characterizes the range of action of the interface force law. We seek the asymptotic form of (13a) for infinitesimal displacement jump $(\lambda_0 - 1)$ requiring that $(\lambda_0 - 1)\rho^{-1}$ remains finite. The limit process that we employ ensures that the resulting infinitesimal strain theory is capable of capturing non-linear decohesive phenomena for interface forces that are short-range. Introduce the variables Λ, Λ_0 defined by $(\lambda - 1)\rho^{-1}$, $(\lambda_0 - 1)\rho^{-1}$, respectively. Then a Taylor series of (13a) about $\rho = 0$ implies that,

$$\sigma - [1 + 2c\rho\Lambda_0]s_r(\Lambda_0) - \tfrac{1}{3}D^2\hat{w}(1)\rho(1 - c)\Lambda_0 + O(\rho^2) = 0, \tag{52}$$

where we have used the fact that there is no residual stress so that $D\hat{w}(1) = 0$. By neglecting the term $O(\rho^2)$ in (52) we obtain the incompressible form of the infinitesimal interface equation for spherically symmetric separations. A similar argument applied to bifurcation condition (45) generates the infinitesimal strain version of that equation,

$$Ds_r(\Lambda_0) = -\frac{\frac{1}{3}\rho(1 - c)D^2\hat{w}(1) + 2c\rho\sigma}{(1 + 2c\rho\Lambda_0)^2}. \tag{53}$$

Thus we have the result that for the infinitesimal strain theory bifurcation always occurs when $Ds_r(\Lambda_0) < 0$, i.e., the interface force is on the descending branch of the force–separation curve. Note that (52) includes a term $2c\rho\Lambda_0 s_r$, small compared to unity, which does not appear in the form that results from a direct derivation from linear theory.

3.4. The Mooney–Rivlin solid

We illustrate the theory for a composite matrix shell consisting of Mooney–Rivlin material which, for deformation (5), may be characterized by,

$$\hat{w}(\lambda) = \frac{\mu}{2}\left[\left(\frac{1}{2} + \beta\right)(\lambda^{-4} + 2\lambda^2 - 3) + \left(\frac{1}{2} - \beta\right)(\lambda^4 + 2\lambda^{-2} - 3)\right], \tag{54}$$

with $\mu > 0$ and $-\frac{1}{2} \leqslant \beta \leqslant \frac{1}{2}$. For this material the pressure function (17) and the stress components (18) can be obtained in the closed form,

$$
\begin{aligned}
\frac{\hat{\pi}}{\mu} &= -\left(\frac{1}{2}-\beta\right)\lambda^4 + (1-2\beta)(\lambda-\lambda_1) - (1+2\beta)(\lambda^{-1}-\lambda_1^{-1}) - \left(\frac{1}{2}-\beta\right)(\lambda^{-2}-\lambda_1^{-2}) + \frac{1}{2}\left(\frac{1}{2}+\beta\right)(\lambda^{-4}+\lambda_1^{-4}) - \sigma\lambda_1^{-2} \\
\frac{T_{rr}}{\mu} &= -(1-2\beta)(\lambda-\lambda_1) + (1+2\beta)(\lambda^{-1}-\lambda_1^{-1}) + \left(\frac{1}{2}-\beta\right)(\lambda^{-2}-\lambda_1^{-2}) + \frac{1}{2}\left(\frac{1}{2}+\beta\right)(\lambda^{-4}-\lambda_1^{-4}) + \sigma\lambda_1^{-2} \\
\frac{T_{\theta\theta}}{\mu} &= \frac{T_{\varphi\varphi}}{\mu} = \frac{T_{rr}}{\mu} + \left(\frac{1}{2}-\beta\right)(\lambda^4-\lambda^{-2}) + \left(\frac{1}{2}+\beta\right)(\lambda^2-\lambda^{-4}),
\end{aligned} \tag{55}
$$

where λ_1 is given by (15). Together with (15) and (16) these expressions give the pressure and stress fields as explicit functions of radial coordinate provided we can determine the interface stretch λ_0 from the interface equation (13a).

For the Mooney–Rivlin strain energy density (54) the interface equation (13a) and the bifurcation condition (45) become,

$$
\begin{aligned}
F(\lambda_0,\sigma) = {} & \sigma_\mu\hat{\lambda}_1^{-2}(\lambda_0) + (1-2\beta)(\lambda_0-\hat{\lambda}_1(\lambda_0)) - \frac{1}{2}\left(\frac{1}{2}+\beta\right)(\lambda_0^{-4}-\hat{\lambda}_1^{-4}(\lambda_0)) - \left(\frac{1}{2}-\beta\right)(\lambda_0^{-2}-\hat{\lambda}_1^{-2}(\lambda_0)) \\
& - (1+2\beta)(\lambda_0^{-1}-\hat{\lambda}_1^{-1}(\lambda_0)) + s_\mu(\lambda_0),
\end{aligned} \tag{56a}
$$

$$
D_{\lambda_0}F(\lambda_0,\sigma) = 0, \tag{56b}
$$

where $\hat{\lambda}_1$ is the function defined by the relation (15). Note that in (56a) and (56b) σ_μ and s_μ are the remote load and interface force normalized with respect to shear modulus μ. The solutions of equations (56a) and (56b) are contained in the set $\{\bar{\lambda},\bar{\sigma}_\mu\}$ with local behavior near these points governed by (47a) and (47b). Note that the sign of the denominator of (47a) and (47b), and ultimately the character of the bifurcation behavior, will depend on a particular bifurcation point $\bar{\lambda}$, $\bar{\sigma}_\mu$ considered.

The interface equation and bifurcation condition governing the limiting cases of the unbounded Mooney–Rivlin matrix and the Mooney–Rivlin matrix shell follow directly from (54), (48) and (49) (unbounded matrix) and (54), (50), (51a), (51b) (thin matrix shell) and will not be presented. Because of their simplicity we do however present the equations of the infinitesimal strain theory to which all hyperelastic matrix material equations must ultimately collapse in the limit of infinitesimal deformations. Eqs. (52) and (53) together with the fact that $D^2\hat{w}(1) = 12\mu$ yields,

$$
\begin{aligned}
&\sigma_\mu - [1+2c\rho\Lambda_0]s_\mu(\Lambda_0) - 4\rho(1-c)\Lambda_0 = 0, \\
&Ds_\mu(\Lambda_0) = -2\rho\frac{2(1-c)+c\sigma_\mu}{(1+2c\rho\Lambda_0)^2},
\end{aligned} \tag{57}
$$

where it is recalled that $\Lambda_0 = (\lambda_0-1)\rho^{-1}$ is defined to be the normalized interface displacement jump and ρ is a characteristic (non-dimensional) length of the interface force law.

Assume interface response is characterized by the simple physically based normal exponential force law of Ferrante et al. (1982), modified to account for interfacial shear,

$$
\begin{aligned}
s_r(\tilde{u},\tilde{v}) &= es_{\max}\left\{\frac{\tilde{u}}{\rho} - \frac{1}{2}\eta\left(\frac{\tilde{v}}{\rho}\right)^2\right\}e^{-\tilde{u}/\rho}, \\
s_\theta(\tilde{u},\tilde{v}) &= es_{\max}\left\{\eta\frac{\tilde{v}}{\rho}\right\}e^{-\tilde{u}/\rho},
\end{aligned} \tag{58}
$$

where the single parameter $\eta(\geqslant 0)$ is a measure of both the shear stiffness of the interface and the strength of the coupling between the normal and tangential separation modes. The quantity $s_{\max}$ is the interface strength, and ρ is a phenomenological force length parameter normalized with respect to inclusion radius R_0

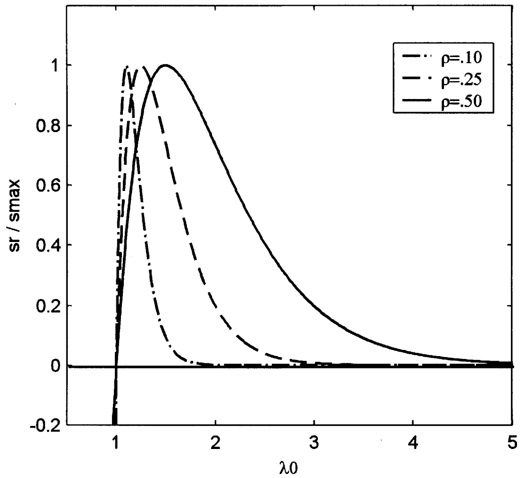

Fig. 2. The interface force law.

(Fig. 2 illustrates interface response in purely normal separation $s_r(\lambda_0 - 1)$). [1] This simple model characterizes non-linear normal separation and linear shear slip consistent with the constraints (35). It, and other more sophisticated models, has been discussed at length in Needleman (1992). Note that the assumed linearity in interfacial shear slip $\tilde{v}$ is appropriate since $\tilde{v}$ vanishes for spherically symmetric equilibrium states and we are concerned with initial bifurcation behavior only.

For this force law the behavior of interface equation (56a) is depicted in Fig. 3, which illustrates the effect of concentration c on the evolution of interface stretch λ_0 with normalized load σ_μ. These curves were drawn for the neo-Hookian matrix ($\beta = 1/2$) and for interface parameter values $s_{\max} = 0.25\mu$, and $\rho = 0.25$. For the first bifurcation point (smallest λ_0) (for $c = 0.75$) the denominator of the first term of (47b) is negative so the bifurcation point ($\bar{\lambda} = 1.4766, \bar{\sigma}_\mu = 0.5975$) is a saddle node of the kind depicted in Fig. 1(b). The second bifurcation point ($\bar{\lambda} = 2.4003, \bar{\sigma}_\mu = 0.4688$) renders the denominator of the first term of (47b) positive so the point is a saddle node depicted in Fig. 1(a). Furthermore, Fig. 3 clearly indicates that increasing the concentration c past a critical value can precipitate bifurcation. For $s_{\max} = 0.25\mu$ and $\rho = 0.25$ that value is obtainable numerically and is given by $c = 0.5038$ as indicated in the figure. Fig. 4 depicts the effect of force length parameter (ρ) on response at fixed concentration ($c = 0.50$). The figure indicates that there is a threshold value of force length parameter ρ above which bifurcation is not possible. For $s_{\max} = 0.25\mu$ that value is obtainable numerically and is given by $\rho = 0.2384$ as indicated in the figure. Fig. 5 depicts curves of normalized circumferential and radial stress at the interface as a function of σ/μ for $\rho = 0.25$ and for two values of volume concentration (c). For $c = 0.50$ the radial stress (equal to the interface traction) at bifurcation abruptly decreases as the interface unloads. This is accompanied by a jump in the circumferential stress. For $c = 0.25$ no such bifurcation occurs and the radial (circumferential) stress gradually decreases (increases). Note that for small values of remote load the response is that of the rigidly bonded linear elastic solution (dotted line in the Fig. 5). The void solution of the linear elastic problem is not obtained as the remote load increases (and the interface unloads) due to the influence of finite strain.

[1] Note that when $\eta = 0$, s_r is independent of $\tilde{v}$. In that case we write the interface force as $s_r(\lambda_0 - 1)$.

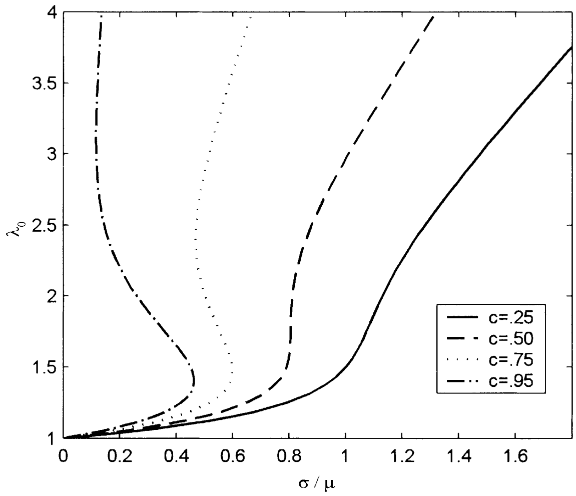

Fig. 3. Interface stretch vs normalized remote load: $s_{\max} = 0.25\mu$, $\rho = 0.25$.

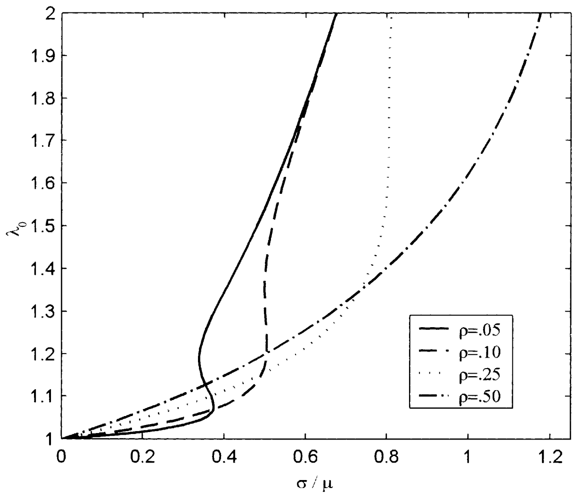

Fig. 4. Interface stretch vs normalized remote load: $s_{\max} = 0.25\mu$, $c = 0.50$.

3.5. The rigid body bifurcation mode

Consider now the more interesting $n = 1$ rigid body mode. Eqs. (27) governing U_1 together with boundary conditions (42a) and (42b) and rigid body equilibrium constraint (38) and (43) yield, after some algebra,

$$\frac{\Gamma(\lambda_0, \sigma)}{\Phi(\lambda_0, \sigma) + c^{5/3} D_{\bar{u}} s_r(\lambda_0 - 1, 0) \Sigma(r_0)} D_{\bar{u}} s_r(\lambda_0 - 1, 0) w_0 = 0, \tag{59a}$$

$$\{\Phi(\lambda_0, \sigma) + c^{5/3} D_{\bar{u}} s_r(\lambda_0 - 1, 0) \Sigma(r_0)\} C_1 = c^{5/3} D_{\bar{u}} s_r(\lambda_0 - 1, 0) w_0, \tag{59b}$$

$$C_2 = \Psi(r_1, \sigma) C_1, \tag{59c}$$

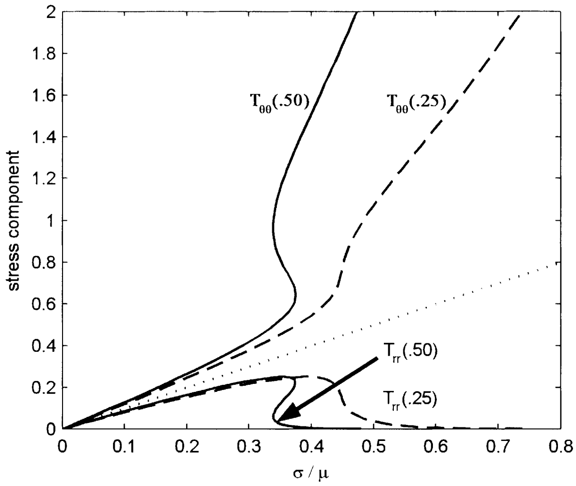

Fig. 5. Normalized radial and circumferential boundary stress: $s_{\max} = 0.25\mu$, $\rho = 0.25$.

where the functions Φ, Γ, Ψ, Σ are defined by,

$$\Phi(\lambda_0,\sigma) = c^{5/3}R_0^2\lambda_0^{-2}(\lambda_0^3-1)^{2/3}D\hat{w}(\lambda_0)\int_{\lambda_1}^{\lambda_0}(\lambda^3-1)^{-2/3}\frac{\lambda^2D^2\hat{w}(\lambda)-\lambda D\hat{w}(\lambda)}{[D\hat{w}(\lambda)]^2}\mathrm{d}\lambda + \frac{2\sigma R_0^2\lambda_1^2\lambda_0^{-2}(\lambda_1^3-1)^2(\lambda_0^3-1)^{-1}D\hat{w}(\lambda_0)}{D\hat{w}(\lambda_1)[\lambda_1^3D\hat{w}(\lambda_1)-2\sigma(\lambda_1^3-1)]}, \tag{60a}$$

$$\Gamma(\lambda_0,\sigma) = -\left[D_{\tilde{u}}s_r(\lambda_0-1,0)+2D_{\tilde{v}}s_\theta(\lambda_0-1,0)\left(1-\frac{\lambda_0^3-1}{D\hat{w}(\lambda_0)}D_{\tilde{u}}s_r(\lambda_0-1,0)\right)\right]\Phi(\lambda_0,\sigma) + D_{\tilde{u}}s_r(\lambda_0-1,0)D_{\tilde{v}}s_\theta(\lambda_0-1,0)\frac{2c^{5/3}R_0^2(\lambda_0^3-1)}{D\hat{w}(\lambda_0)}. \tag{60b}$$

$$\Psi(r_1,\sigma) = \left[\frac{2\lambda_1^5(\lambda_1^3-1)R_1^5}{\lambda_1^3D\hat{w}(\lambda_1)-2\sigma(\lambda_1^3-1)} - 2\int^{r_1}t^4\frac{\lambda^3-\lambda^{-3}}{D\hat{w}(\lambda)}\mathrm{d}t\right], \tag{60c}$$

$$\Sigma(r_0,r_1,\sigma) = 2\int^{r_0}\lambda(z)z^{-4}\left[\int^{z}t^4\frac{\lambda^3(t)-\lambda^{-3}(t)}{D\hat{w}(\lambda(t))}\mathrm{d}t\right]\mathrm{d}z + \Psi(r_1,\sigma)\int^{r_0}\lambda(z)z^{-4}\mathrm{d}z. \tag{60d}$$

Eqs. (44) and (59a)–(59c) govern spherically symmetric and non-symmetric bifurcation arising from non-linear interface characterization as well as from non-linear matrix response at finite strain.

The solution to the problem of the pressurized spherical shell treated by Ogden (1984) is recovered in the present analysis by assuming there is no load applied to the outer boundary while the loading on the inner boundary is a constant pressure P. In boundary conditions (42a) and (42b) this amounts to setting the dead loading σ, as well as the integral on the right-hand side of (42b), to zero. Then from (60a) and (59b) the bifurcation condition becomes,

$$\Phi(\lambda_0,0)C_1=0,$$
$$\Phi(\lambda_0,0)=c^{5/3}R_0^2\lambda_0^{-2}(\lambda_0^3-1)^{2/3}D\hat{w}(\lambda_0)\int_{\lambda_1}^{\lambda_0}(\lambda^3-1)^{-2/3}\frac{\lambda^2D^2\hat{w}(\lambda)-\lambda D\hat{w}(\lambda)}{[D\hat{w}(\lambda)]^2}\mathrm{d}\lambda. \tag{61}$$

Bifurcation to an $n=1$ mode is possible when $\Phi(\lambda_0,0)=0$, otherwise C_1 and C_2 (by (59c)), vanish. Note that in this case bifurcation signals a transition in deformation mode arising from bulk matrix response characterized by strain energy density $\hat{w}$.

The spherically symmetric eigenmode has been considered in the previous section. Consider the non-symmetric eigenmode governed by (59a)–(59c). Bifurcation to a rigid body mode requires the satisfaction of (59a) for $w_0\neq 0$. This is possible when,

$$\begin{aligned}0&=\Gamma(\lambda_0,\sigma)\\&=-\left[D_{\tilde{u}}s_r(\lambda_0-1,0)+2D_{\tilde{v}}s_\theta(\lambda_0-1,0)\left(1-\frac{\lambda_0^3-1}{D\hat{w}(\lambda_0)}D_{\tilde{u}}s_r(\lambda_0-1,0)\right)\right]\Phi(\lambda_0,\sigma)\\&\quad+D_{\tilde{u}}s_r(\lambda_0-1,0)D_{\tilde{v}}s_\theta(\lambda_0-1,0)\frac{2c^{5/3}R_0^2(\lambda_0^3-1)}{D\hat{w}(\lambda_0)}.\end{aligned} \tag{62}$$

Note that $D_{\tilde{u}}s_r(\lambda_0-1,0)$ will not generally vanish unless the constitutive characteristics of the interface are such that the interface cannot support shear, i.e., $s_\theta=0$. This case will be treated separately. Condition (62) involves both λ_0 and σ and therefore must be coupled to (13a) in order to obtain the critical values of stretch and load at bifurcation. By (59b) and (59c) C_1 and C_2 do not vanish so the matrix shell has a non-vanishing deformation mode proportional to $U_1(r)P_1(\cos\theta)$, $V_1(r)\dot{P}_1(\cos\theta)$.

The inhomogeneous problem, corresponding to continuing spherically symmetric equilibrium with $\Delta\sigma\neq 0$ and $w_0=0$, is governed by (44). In order for solutions to exist the coefficient of U_0/r_0 cannot vanish, i.e., (45) cannot hold. The (Fredholm) alternative is that (45) holds and there are non-trivial solutions to the homogeneous problem but then solutions to the inhomogeneous problem do not exist. A non-symmetric solution associated with (44) requires that $\Delta\sigma\neq 0$ and $w_0\neq 0$. Here the eigenmode with multipliers $U_1(r)$, $V_1(r)$ is orthogonal to the incremental loading as required. In what follows, we provide explicit results for the case of the composite sphere with smooth interface, and the case of the unbounded matrix with general interface constitutive characteristics.

3.5.1. The smooth interface ($s_\theta=0$, s_r independent of u_θ)

When the interface is smooth, i.e., it cannot support shear, the bifurcation conditions simplify considerably so that (59a) is satisfied for $w_0\neq 0$ when,

$$D_{\tilde{u}}s_r(\lambda_0-1)=0. \tag{63}$$

This condition simply states that the rigid body bifurcation mode may occur when the interface force attains its maximum value. Note that for the smooth interface, the composite shell will have vanishing deformation modes $U_1(r)P_1(\cos\theta)$, $V_1(r)\dot{P}_1(\cos\theta)$. This follows directly from (59b) and (59c) where C_1, C_2 and therefore V_1, U_1 are zero.

Now assume that the non-linear normal interface force–separation relation $s_r(\lambda_0-1)$ takes its maximum value $s_{\max}$ at $\lambda_0=1+\rho$ where $s_{\max}>0$ and $\rho>0$ (e.g. (58) with $\tilde{u}=\lambda_0-1$, $\eta=0$). Then, $s_r(\rho)=s_{\max}$, $D_{\lambda_0}s_r(\rho)=0$. Thus, from (63) bifurcation occurs when the interface force attains its maximum value $s_{\max}$ which occurs at an interface stretch $\lambda_0=1+\rho$. If we further assume that the matrix is unbounded we have the result that, for the smooth interface, non-symmetric bifurcation always precedes spherically symmetric bifurcation. This follows from the fact that the interface force–separation relation has, by assumption, a single absolute maximum on $\lambda_0\in[1,\infty]$ and, for spherically symmetric bifurcation $D_{\lambda_0}s_r(\lambda_0-1)<0$ (recall (49)). For the finite matrix shell we have shown that spherically symmetric bifurcation may occur when

$D_{\lambda_0} s_r(\lambda_0 - 1)$ is positive or negative so no general statement can be made regarding the order of bifurcation. Note that, unlike spherically symmetric bifurcation, there is no threshold value of ρ for which bifurcation will not occur. Now recall (13a). For the finite matrix shell the critical load at bifurcation is,

$$\sigma = [1 + c((1+\rho)^3 - 1)]^{2/3} \left\{ s_{\max} + \int_{[1+c((1+\rho)^3-1)]^{1/3}}^{1+\rho} \frac{D\hat{w}(\lambda)}{(\lambda^3 - 1)} \mathrm{d}\lambda \right\}, \tag{64}$$

and for the unbounded matrix ($c = 0$) we have the simple result,

$$\sigma = s_{\max} + \int_1^{1+\rho} \frac{D\hat{w}(\lambda)}{(\lambda^3 - 1)} \mathrm{d}\lambda. \tag{65}$$

Now $[1 + c((1+\rho)^3 - 1)]^{1/3} \in (1, 1+\rho)$ for $c \in (0,1)$ and $\rho > 0$. Then by the Baker–Ericksen inequality (29) the critical load (64) (or (65)) always exceeds the interface strength $s_{\max}$. Note that the critical load at bifurcation depends only on the matrix strain energy density, the volume concentration, the maximum value of interface force, and the interface stretch at which this is attained. For a matrix modeled as a Mooney–Rivlin solid with strain energy density given by (54), the critical load relation (64) (and (65)) may be integrated exactly although the result is cumbersome and therefore shown only in graph form. Fig. 6 is a plot of critical load (normalized with respect to matrix shear modulus) versus interface force length parameter at various values of concentration for $s_{\max} = 0.25\mu$ and $\beta = 1/2$ (neo-Hookian solid). As expected for a fixed concentration the critical load increases with increasing force length parameter. Also, the curves indicate that at a fixed force length parameter the critical load decreases with increasing value of concentration. This last result is not generally true and is a consequence of the relative values of interface strength and matrix shear modulus. For example for interface strength equal to the shear modulus increasing concentration increases the critical load for a range of values of force length parameter (ρ).

We record here the critical load for an infinitesimal strain non-linear theory. Let $\lambda = \rho\Lambda + 1$ and expand (64) in a series about $\rho = 0$ keeping only linear terms in ρ. Then,

$$\sigma = (1 + 2c\rho)s_{\max} + \frac{1}{3}(1 - c)D^2\hat{w}(1)\rho. \tag{66}$$

The effect of concentration in this formula is transparent and is clearly influenced by the relative magnitudes of interface strength $s_{\max}$ and matrix shear modulus (proportional to $D^2\hat{w}(1)$).

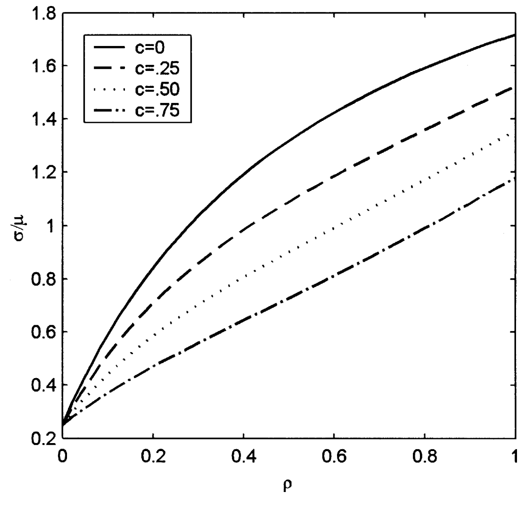

Fig. 6. Critical load vs force length parameter: smooth interface, $s_{\max} = 0.25\mu$.

3.5.2. The unbounded matrix ($c = 0$)

Eqs. (59a)–(59c), (60a)–(60d) govern the non-symmetric eigenmode for the finite matrix shell assuming that $c \in (0, 1)$. The equations governing the unbounded matrix follow by setting $c = 0$ (or $r_1 = \lambda_1 R_1 \uparrow \infty$). More specifically, (59a) becomes,

$$\left\{ D_{\bar{u}} s_r(\lambda_0 - 1, 0) + 2D_{\bar{v}} s_\theta(\lambda_0 - 1, 0) \left[1 - \frac{\lambda_0^3 - 1}{D\hat{w}(\lambda_0)} D_{\bar{u}} s_r(\lambda_0 - 1, 0) \right] \right\} D_{\bar{u}} s_r(\lambda_0 - 1, 0) w_0 = 0, \tag{67}$$

while (59b) implies that $C_1 = 0$. However, (59c) does not imply that C_2 is necessarily zero. This is because the function Ψ becomes unbounded as r_1 becomes infinite. If $D_{\bar{u}} s_r(\lambda_0 - 1, 0)$ is infinitesimal, then (67) implies that to an infinitesimal quantity squared, $[D_{\bar{u}} s_r(\lambda_0 - 1, 0) + 2D_{\bar{u}} s_r(\lambda_0 - 1, 0)] D_{\bar{u}} s_r(\lambda_0 - 1, 0) w_0 = 0$. In that case $U_1(r_0) = V_1(r_0) = 0$ by (38) and (43). Then by (25b) and (27) it follows that $C_2 = 0$ and $U_1(r) = V_1(r) = 0$. For the unbounded matrix then, bifurcation to a rigid body mode is possible, with $U_1 = V_1 = 0$ and $w_0 \neq 0$, when,

$$0 = D_{\bar{u}} s_r(\lambda_0 - 1, 0) + 2D_{\bar{v}} s_\theta(\lambda_0 - 1, 0). \tag{68}$$

The critical load and critical interface stretch at bifurcation are obtained below.

Interface force law (58) together with bifurcation condition (68) may be solved to yield the critical value of interface stretch λ_0 at bifurcation,

$$\lambda_0 = 1 + \rho(1 + 2\eta). \tag{69}$$

The critical load at bifurcation then follows from (48),

$$\sigma = s_r(\rho(1 + 2\eta)) + \int_1^{1+\rho(1+2\eta)} \frac{D\hat{w}(\lambda)}{(\lambda^3 - 1)} \mathrm{d}\lambda. \tag{70}$$

Note that, because $\rho, \eta \geqslant 0$ and $D\hat{w}(\lambda)(\lambda^3 - 1)^{-1} > 0$ for $\lambda \geqslant 1$, the quantity $s_r(\rho(1 + 2\eta))$ forms a lower bound to the critical load.

The critical load for an infinitesimal strain non-linear theory follows by substituting $\lambda = \rho\Lambda + 1$ into (70) and expanding in a series about $\rho = 0$ keeping only linear terms in ρ. The critical load in this case is, $\sigma = (1 + 2\eta) s_{\max} \mathrm{e}^{-2\eta} + (1 + 2\eta) D^2\hat{w}(1)\rho/3$.

For the Mooney–Rivlin matrix with strain-energy density given by (54) the critical load may be obtained in closed form from (70). The result is shown graphically in Fig. 7, which is a plot of critical load (normalized with respect to shear modulus) versus force length parameter for various values of interfacial shear stiffness parameter and for $s_{\max} = 0.25\mu$ and $\beta = 1/2$ (neo-Hookian solid). The plot seems to indicate that, at fixed force length, the critical load increases with increasing shear stiffness, which is physically reasonable. However, in Fig. 8 we show a blow up of the same plot for a smaller interval of force length parameter. This plot indicates the paradoxical result that, for small force length parameter, the critical load decreases with increasing interfacial shear stiffness. In this case it turns out that for a range of values of interface shear stiffness parameter, the critical load for non-symmetrical bifurcation lies on a branch of equilibrium states that is unreachable by continuous increase in load from the undeformed state. When this occurs, the first bifurcation point will be spherically symmetric. Note that the smallest critical interface stretch λ_0, from (69), always increases with increasing shear stiffness.

The bifurcation domains in η–ρ parameter space may be obtained as in Levy (2001), by simply equating the critical loads for non-symmetrical bifurcation to that of the critical load for symmetrical bifurcation. Thus we solve (48), (49) and (70) for the curve $\hat{\rho}(\eta)$ which defines the boundary between symmetric and non-symmetric bifurcation. For $s_{\max} = 0.25\mu$ and $\beta = 1/2$ (neo-Hookian solid) the results, which are qualitatively the same as the planar problem, are shown in Fig. 9(a). In order to explain the behavior in each domain represented in the figure we utilize the following notation defined in Fig. 9(b). A horizontal

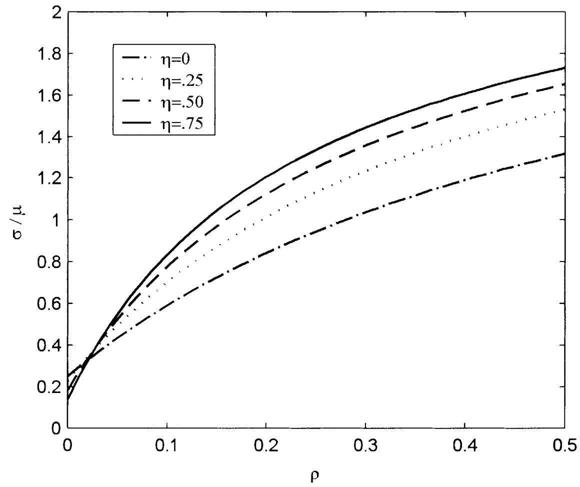

Fig. 7. Critical load vs force length parameter: unbounded matrix, $s_{\max} = 0.25\mu$.

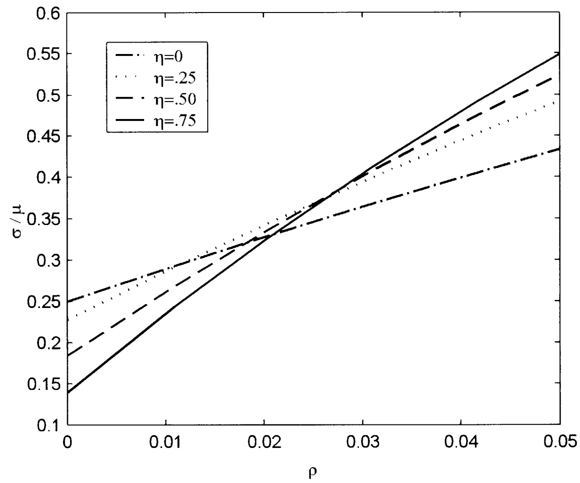

Fig. 8. Critical load vs force length parameter: unbounded matrix, $s_{\max} = 0.25\mu$.

line represents the spherically symmetric response projected on the σ/μ axis so that the remote load ratio as well as spherically symmetric equilibrium states lies on this line. The circles designate spherically symmetric bifurcation points, which generally exist in pairs. Note that, as shown in Fig. 9(b), the second spherically symmetric bifurcation point is unreachable by continuous increase in load from the undeformed state although it occurs at a smaller critical load then the first point. Consider now Fig. 9(a). Branches of non-symmetric equilibria (accompanied by rigid inclusion displacement) are indicated by a short vertical line that initiates from a non-symmetric bifurcation point denoted by a cross. An open circle indicates a spherically symmetric bifurcation point on the current branch. A circle with the horizontal line passing through it situated to the left (right) of the open circle indicates that the spherically symmetric bifurcation point lies on a branch above (below) the current one. The solid line divides the domains in which non-symmetric bifurcation and spherically symmetric bifurcation dominate. The horizontal line (solid and

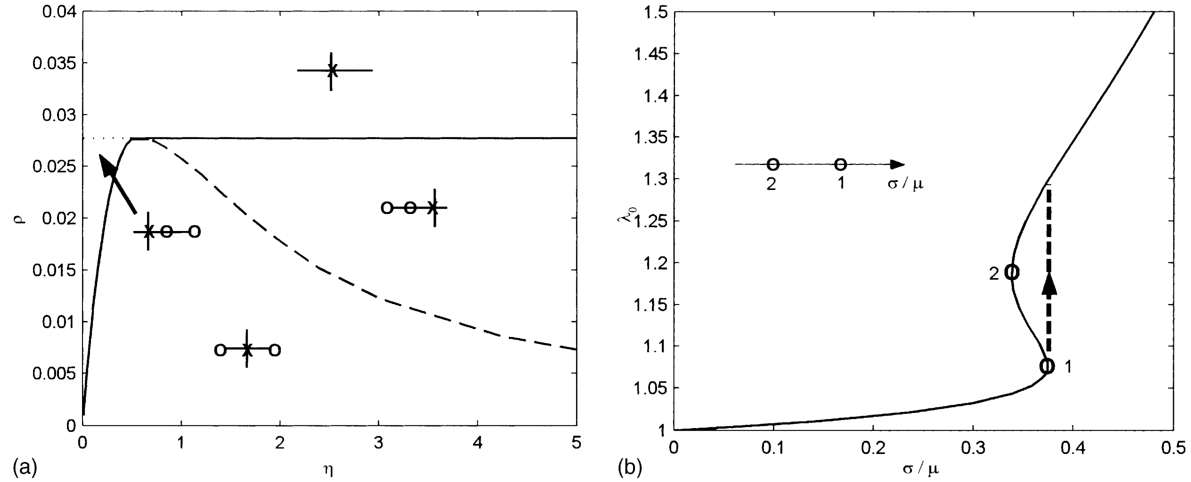

Fig. 9. (a) Parameter regions of the first bifurcation point: Mooney–Rivlin, $s_{\max} = 0.25\mu$ (spherically symmetric bifurcation point O; non-symmetric bifurcation point X); (b) notation.

dotted) indicates the threshold above which spherically symmetric bifurcation cannot occur. The dashed line of the figure is a solution to the equations arising from equating the unreachable "symmetrical" critical load to the "non-symmetrical" critical load. Note that Fig. 9(a) is incomplete in the sense that global non-symmetrical behavior is not represented owing to the fact that our analysis is based on infinitesimal non-symmetric strain superposed on a given finite spherically symmetric deformation. In order to predict the criticality of the pitchfork bifurcations initiating from non-symmetric points at least one more term would need to be taken in the superposed non-symmetrical strain. Two possibilities, which correspond to the second and first order phase transformations described by Ericksen (1991), are indicated in Fig. 10(a), (b) (note that the horizontal lines in the figures correspond to equilibrium states also). The first characterizes the gradual rigid displacement of the inclusion while the second describes an abrupt movement. Even if the

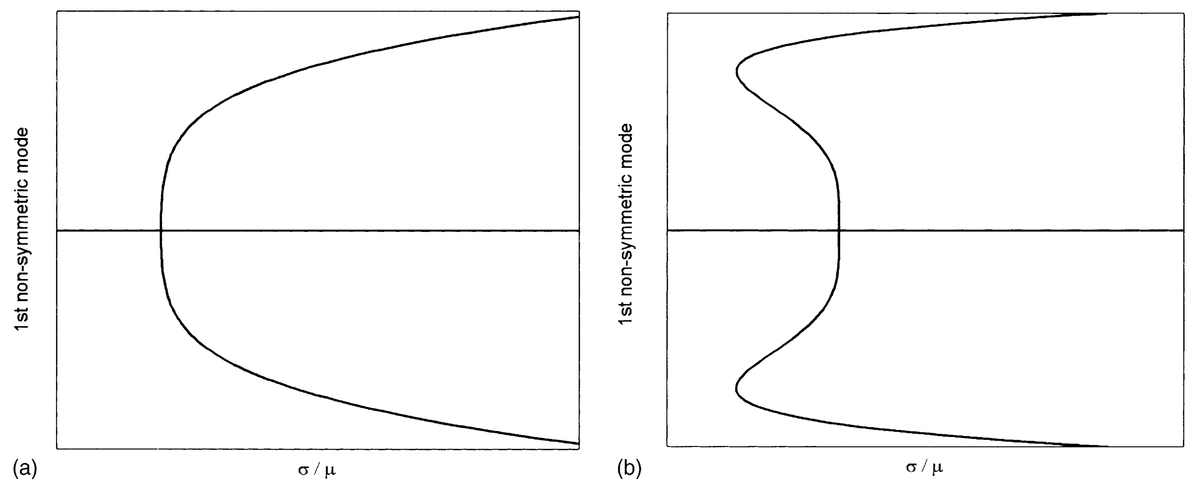

Fig. 10. (a) Global portrait near supercritical pitchfork bifurcation point. Mode vs bifurcation parameter; (b) global portrait near subcritical pitchfork bifurcation point. Mode vs bifurcation parameter.

local behavior near the bifurcation points is completely known the behavior in certain cases may still be indeterminate. As an example consider systems whose (η, ρ) parameters lie in the domain characterized by a non-symmetric bifurcation point which resides between two symmetric points. Clearly, a complete picture of the behavior is possible only with a detailed study of the global structure of branches of equilibria. For a detailed analysis of global issues of this sort, in the context of infinitesimal plane strain, see Levy (1998).

4. Closure

This paper has examined aspects of cavity formation by interfacial separation in a radially loaded composite sphere deforming at finite strain. In particular, such issues as symmetrical/non-symmetrical bifurcation and gradual (ductile) and abrupt (brittle) decohesion have been addressed. Taken together the results presented indicate the influence on the separation process of factors such as concentration, interface force length parameter (or interface energy) and interfacial shear stiffness. For small values of interface force length parameter, finite strain issues will have minimal effect on spherically symmetric bifurcation (and non-symmetrical bifurcation) [2] but not necessarily on the post bifurcation cavity growth process. However, in materials deforming non-linearly at infinitesimal strain quantities such as yield strength and strain hardening exponent can be expected to significantly affect bifurcation and subsequent cavity growth (Levy, 2002). Furthermore, factors such as non-uniformities in interface strength, slight departures from sphericity, or even the existence of remotely situated inhomogeneities, will play a critical roll by biasing the cavity formation process in a particular direction (this has been demonstrated by Levy and Hardikar (1999) in a planar, infinitesimal strain setting). This feature of cavity formation by inclusion debonding underscores the inherent difficulty in predicting cavity shapes for even the simplest of geometries and loadings. The range of validity of analyses of cavity formation at inhomogeneities based on a priori enforced symmetry constraints will therefore depend on factors such as interfacial shear stiffness which, as shown in this paper, can delay non-symmetrical bifurcation but never eliminate it.

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[2] Provided the interfacial shear stiffness is not too large.

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